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SHORT NOTES

The theory of shear stress and shear strain on planes inclined to the principal directions

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Abstract—Given the magnitudes and orientations of the principal stresses, the normal and shear components of stress acting on a plane of arbitrary orientation are determined either graphically or with the aid of a novel type of vector product. The normal stress $\boldsymbol{\sigma}_n$ due to the action of a stress vector $\boldsymbol{\sigma}$ on a plane of pole **n** is the star product $\mathbf{n}*\boldsymbol{\sigma}$, which is defined as a vector oriented parallel to **n** with magnitude equal to the dot product $\mathbf{n}\cdot\boldsymbol{\sigma}$. The shear stress vector $\boldsymbol{\sigma}_s$ is given by the vector difference $\boldsymbol{\sigma} - \boldsymbol{\sigma}_n$. In the case of a deformation, given the magnitudes and orientations of the principal stretches, the shear strain of an arbitrary line is determined by a related, but significantly different, procedure.

INTRODUCTION

LISLE (1989), Means (1989) and Ragan (1990, this issue) draw attention to the problem of determining shear stress $\boldsymbol{\sigma}_s$ on a plane of pole **n** in a general orientation, given the magnitudes and orientations of the principal stresses, $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$. Lisle obtains the direction of shear stress from a construction involving the circular sections of an ellipsoid related to the stress ellipsoid. Means's construction involves rotation of directional data on a stereonet followed by numerical computation; it yields additional information, namely the magnitude of the shear stress, in addition to its direction. In both cases, about as much calculation is required for setting up the construction as is involved in a classical algebraic solution of the problem as reviewed by Ragan (1990, this issue). The constructions are intended to give students an additional understanding of spatial relations. Students vary in their preference for graphical vs algebraic treatments and are advised to use whichever solution they find the most intuitive. In this paper, I present an alternative approach which yields the magnitudes, trends and plunges of the stress vector and its normal and shear components, using either an orientation net or a simple computer/pocket-calculator program. The approach adopted here highlights intrinsic differences between the analysis of shear stress (Zizicas 1955) and shear strain (Treagus 1986), for which different methods of derivation are necessary. It also overcomes problems that arise in previously published techniques when one or more principal stress magnitude is zero or negative.

Ragan (1990, this issue) has divided the problem into two cases; the special case where the principal stress vectors are in the directions of the chosen reference axes, and the general case where they are oblique. In the following discussion, the special case is treated graphically and the general case algebraically.

SHEAR STRAIN VS SHEAR STRESS

Figure 1(a) shows a unit sphere and a surface normal, **n**, extending from the tip of an arbitrarily chosen radius. In Fig. 1(b) the ellipsoid with principal axes S_1, S_2 and S_3 is taken to represent an irrotational deformation of the unit sphere. The length of the chosen radius represents the stretch S of the initial direction **n**, and the deflection of the deformed direction **n′** from the ellipsoid's surface normal **n″** represents the amount of shear strain. A deck of cards stacked on top of the sphere at **n** would be rotated during deformation and simultaneously sheared through the angle $\mathbf{n}'\wedge\mathbf{n}''$; the shear direction is measured from **n″** towards **n′** and it is generally oblique to all three principal planes.

In contrast to Fig. 1(a), the sphere in Fig. 1(c) represents the set of poles to planes of all possible orientations and the ellipsoid of Fig. 1(d) represents a state of triaxial stress with principal stresses $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$. The same radius, relabelled $\boldsymbol{\sigma}$, now represents the stress vector acting on the plane of pole **n**. The stress ellipsoid's tangent plane at the radius $\boldsymbol{\sigma}$ is not shown because it is the plane of conjugate stresses and is not related to the plane of pole **n** on which $\boldsymbol{\sigma}$ acts (the plane illustrated in Fig. 1c). The components of $\boldsymbol{\sigma}$ resolved perpendicular and parallel to the plane on which it acts are the normal and shear stress components, $\boldsymbol{\sigma}_n$ and $\boldsymbol{\sigma}_s$. Thus, shear stress and shear strain differ both in magnitude and direction even when the stress and strain states are

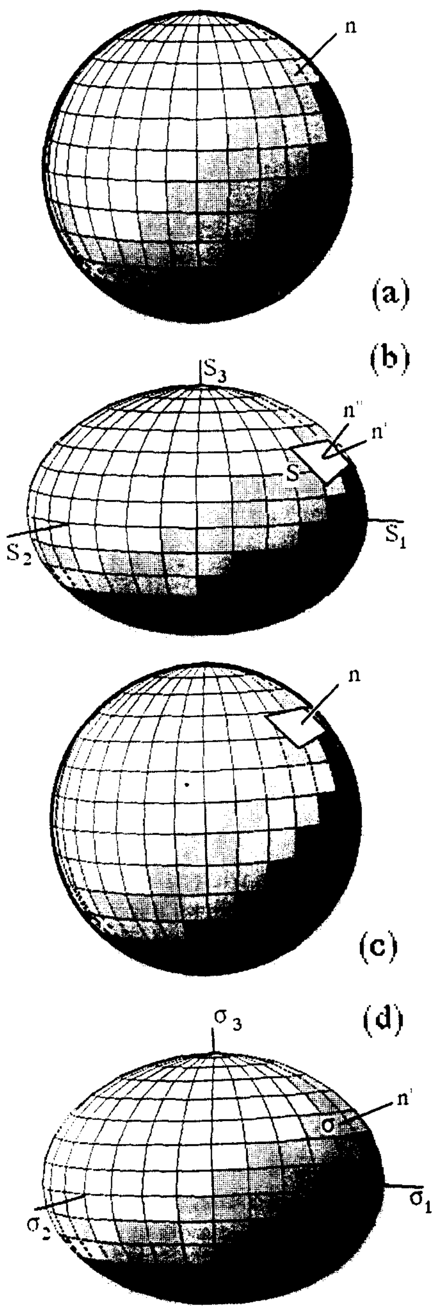

Fig. 1. (a) Unit reference sphere with arbitrary surface normal labelled **n**. (b) Strain ellipsoid with deformed surface normal labelled **n**″ and radius vector S (extended along line **n**′). Principal stretches are labelled S_1, S_2, S_3. (c) Unit reference sphere with pole to arbitrary tangent plane marked **n**. (d) Stress ellipsoid with principal stresses labelled σ_1, σ_2, σ_3. The stress acting on the plane in (c) is labelled σ. Its line of action is **n**′.

represented by identical ellipsoids; similarly, normal stress and longitudinal strain are two entirely different quantities. A problem arises in visualizing the stress components because the stress vector and the plane on which it acts do not appear in the same figure.

GRAPHICAL ANALYSIS OF STRESS

To facilitate graphical determination of normal and shear stress components it is necessary to transfer the stress vector and the plane on which it acts onto one diagram. This is achieved on the orthographic orientation net of Fig. 2(a). Three principal sections of the stress ellipsoid are constructed representing an architect's plan, side and elevation view (see inset). Because of symmetry, it is sufficient to represent only one quadrant of each section and two sections suffice in practice; the third is added for the sake of clarity but is not used in the following construction. The quadrants of the principal elliptical sections are constructed on an overlay by marking $\boldsymbol{\sigma}_1$ along the net's North axis and then drawing $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$ to scale along the East and Down directions. For example, let $\boldsymbol{\sigma}_1 = 10$ MPa, $\boldsymbol{\sigma}_2 = 5$ MPa, $\boldsymbol{\sigma}_3 = 3$ MPa (MPa = MegaPascals), and let the orientation net have a radius of 10 cm (as in De Paor 1983). Then $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$

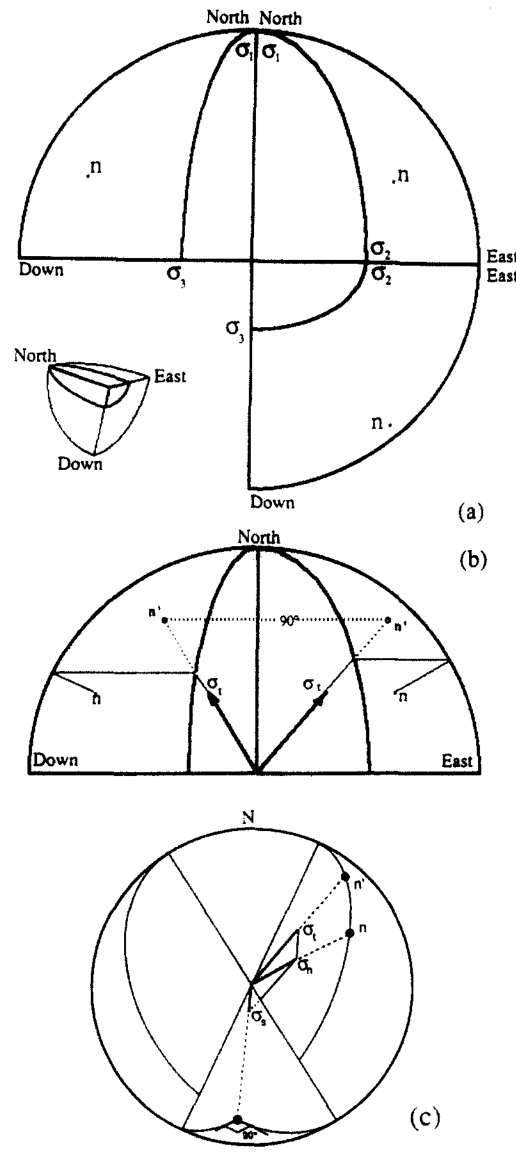

Fig. 2. (a) Graphical construction for representing the three principal sections of the stress ellipsoid when it is oriented parallel to the reference directions. A perspective view is inset. The three points labelled **n** are three different projections of the pole to a chosen plane. See text for explanation. (b) Construction for determining the total stress $\boldsymbol{\sigma}_t$ acting on the plane of pole **n**. Black dots mark the orientation **n**′ of $\boldsymbol{\sigma}_t$ in each quadrant. (c) Graphical resolution of total stress $\boldsymbol{\sigma}_t$ into normal and shear components, $\boldsymbol{\sigma}_n$ and $\boldsymbol{\sigma}_s$. A great circle is constructed through the directions **n**′ and **n** and extended to intersect the plane of pole **n**. The parallelogram rule is used to determine normal and shear stress magnitudes projected in the directions indicated by the black dots.

are represented by lines 10, 5 and 3 cm long, respectively. Elliptical great circles of the orientation net serve to guide the drawing of two elliptical traces (the $\sigma_2 - \sigma_3$ elliptical trace may be added freehand as it is not used in the construction). The pole **n** is plotted at, say, (trend = 060°, plunge = 45°) in the North–East quadrant, then transferred 90° along a small circle to the equivalent North–Down projection. The stress vector acting on the plane of pole **n** is found using the construction in Fig. 2(b), as follows: using both the North–East and North–Down quadrants, trace the direction of **n** radially out to the perimeter of the net, along a small circle to the stress ellipse, and then radially in to the small circle you started on. This yields two projected views of the total stress vector σ_t whose tip is located in the interior space of the reference sphere at Cartesian co-ordinates {3.4N, 2.6E, 2D} (on a 10 cm net). To find the point where this vector pierces the sphere (its "orientation point", De Paor 1979), use the dotted construction lines in Fig. 2(b) as follows: extend the lines of the two projections of σ_t until two points are found to lie 90° apart on a common small circle. These points are the only two that represent the stress vector's orientation compatibly in the two quadrants. Since they represent the same direction, they are both labelled **n**′ and the trend and plunge is measured as {042°, 24°} in the North–East quadrant. The stress magnitude is given by the ratio of the distance from the center of the net to points σ_t and **n**′ (4.7 cm/9.1 cm yields 0.52 times 10 MPa, or 5.2 MPa in this example). Finally, σ_t may be resolved into normal and shear components, σ_n and σ_s, using the parallelogram law as in Fig. 2(c), which gives

$$\sigma_t = \{5.2 \text{ MPa } (=4.7 \text{ cm}/9.1 \text{ cm}), 042°, 24°\}$$
$$\sigma_n = \{4.6 \text{ MPa } (=3.2 \text{ cm}/7 \text{ cm}), 060°, 45°\}$$
$$\sigma_s = \{2.2 \text{ MPa } (=1.9 \text{ cm}/8.6 \text{ cm}), 188°, 32°\}.$$

The above construction works even when one or more principal stress is of zero magnitude. When a principal stress is negative, the geographical labels must be adjusted. For example, if the N–S directed stress σ_1 is negative (tensile) but larger in magnitude than σ_2 and σ_3 which are positive (compressive), the stress vector is interpreted to lie in the South–East quadrant when the plane's pole **n** lies in the North–East quadrant(and vice versa: similarly for other cases). Note that, contrary to textbook wisdom, the stress state is represented by an ellipsoid even when the principal stresses differ in sign (De Paor 1981, 1983).

THE STAR PRODUCT

Whilst Figs. 1 and 2 may help one to understand the problem an algebraic solution using a micro-computer or pocket-calculator is more efficient, especially if the principal stresses are oblique to the reference axes. In order to simplify the algebraic presentation a novel type of vector product is employed. Most students are familiar with the two basic methods of multiplying two

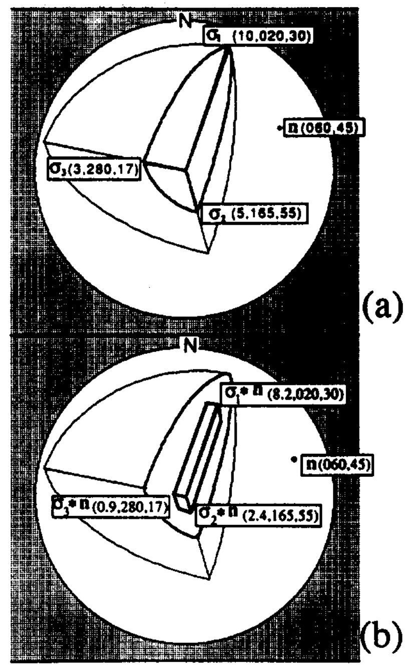

Fig. 3. (a) General case of a stress ellipsoid oriented oblique to the reference axes. The data chosen for the numerical example in the text is shown in (magnitude, trend, plunge) format. (b) Graphical illustration of the star products in equation (1). See text for details.

vectors (say **a** = {1, 2, 3} and **b** = {4, 5, 6}), namely the dot product **a** · **b** and the cross product **a** × **b**. A vector product, called the "star product" **a*b** is here defined as a *vector* parallel to **a** with a magnitude equal to the dot product **a** · **b**. Since **a** · **b** = (1)(4) + (2)(5) + (3)(6) = 32, **a*b** = {8.6, 17.2, 25.8} in this case. Familiarity with the star product may take some practice, but it is worthwhile as it obviates the need to introduce tensor algebra in the general derivations that follow.

ALGEBRAIC ANALYSIS OF STRESS

Given the principal stresses σ_1, σ_2 and σ_3 of (Fig. 3a), the simplest mathematical representation of the total stress σ_t on a plane of pole **n** is

$$\sigma_t = \sigma_1 * \mathbf{n} + \sigma_2 * \mathbf{n} + \sigma_3 * \mathbf{n}. \quad (1)$$

(Note that σ_1, σ_2, and σ_3 are *not* oriented along the reference axes. Each is a vector with three non-zero components.) The form of equation (1) highlights the fact that each principal stress contributes to the total stress a vector component oriented along the principal direction (Fig. 3b) with a magnitude decreased by a factor that depends on its obliquity to the plane's pole **n**. Each star product is represented by a side of the rectilinear box in Fig. 3(b). The vector sum, σ_t is given by the leading diagonal of the box (not shown), which extends from the center to the surface of the stress ellipsoid. The normal component of stress σ_n is given by projecting the stress vector σ_t onto the direction of the normal vector **n**. Since **n** is of unit magnitude, one may write

$$\sigma_n = \mathbf{n} * \sigma_t. \quad (2)$$

The shear component is simply the vector difference,

$$\sigma_s = \sigma_t - \sigma_n. \quad (3)$$

For the data of Fig. 3(a), the first step is to convert from polar (length, trend, plunge) co-ordinates to Cartesian {North, East, Down} co-ordinates,

$$\mathbf{n} = (1, 060°, 45°) = \{0.35N, 0.61E, 0.70D\}$$
$$\sigma_1 = (10, 020°, 30°) = \{8.14N, 2.96E, 0.70D\}$$
$$\sigma_2 = (5, 165°, 55°) = \{-2.77N, 0.74E, 4.10D\}$$
$$\sigma_3 = (3, 280°, 17°) = \{0.50N, -2.82E, 0.88D\}.$$

Then the star products are calculated

$$\sigma_1 * \mathbf{n} = \{6.70N, 2.44E, 4.11D\} = (8.23, 020°, 30°)$$
$$\sigma_2 * \mathbf{n} = \{-1.31N, 0.35E, 1.94D\} = (2.37, 165°, 55°)$$
$$\sigma_3 * \mathbf{n} = \{-0.16N, 0.88E, -0.27D\} = (0.93, 280°, 17°)$$

and their vector sum is obtained simply by adding Cartesian coefficients,

$$\sigma_t = \{5.23N, 3.67E, 5.78D\} = (8.62, 035°, 42°).$$

A BASIC micro-computer program to solve equations (1)–(3) is presented in the Appendix. In addition to solving these equations, the code deals with the conversion of data from the (magnitude, trend°, plunge°) format used in structural geology to the {North, East, Down} Cartesian co-ordinate format of vector algebra, and vice versa. The actual algorithm for solving the equations occupies only a small block of code and may be readily adapted for pocket-calculator use.

GRAPHICAL ANALYSIS OF SHEAR STRAIN

The case of strain differs from that of stress as illustrated in Fig. 1 (see also Treagus 1986). However, the stretch of a line is directly equivalent to the total stress vector and does not require a separate treatment. Furthermore, it is a simple matter to determine shear strain by adapting previous constructions for the special case where ellipsoid axes parallel reference axes (Fig. 4). Let (t, p) be the trend and plunge of an arbitrary line **n**. Since we are not concerned with longitudinal strains, the construction in Fig. 2(b) may be short-circuited to yield only the direction (t', p') of the deformed line **n**′. Using both the North–East and North–Down quadrants, one simply traces zig-zag paths as in Fig. 2(b) radially out to the perimeter, in to the strain ellipses, and then out to points **n**′ with trend and plunge (t', p') that are 90° apart on a common small circle.

March (1932) demonstrated that when a plane undergoes deformation its pole behaves as if it were a line undergoing reverse deformation. Therefore, to locate the pole to the deformed plane whose initial pole was **n**, one must reverse the construction of Fig. 2(b). That is, one traces radially in from **n** to the ellipse, out along a small circle to the perimeter, and radially back to two equivalent points (t'', p'') separated by 90° along a small circle in the North–East and North–Down quadrants. These represent the pole $\mathbf{n}'' = (t'', p'')$. The angular shear ψ of the initial line **n** is the angle between **n**′ and **n**″ (note that **n**, **n**′ and **n**″ are not generally coplanar). ψ can be measured using the standard orientation net method or the dot product method (below) and its tangent determined to yield the required shear strain γ.

Often in structural geology, one knows the final orientation of a line, not its initial orientation. In that event, **n**′ is given and the reverse deformation construction is applied twice to yield first **n** and then **n**″.

ALGEBRAIC ANALYSES OF SHEAR STRAIN

Again it is simplest to solve the problem algebraically, especially when the principal directions are not parallel to the reference axes. Given the principal stretch vectors S_1, S_2, S_3 and the initial pole **n** with trend t and plunge p, these four vectors are first converted from polar to Cartesian co-ordinates and then substituted in equation (1) to yield the stretch vector S,

$$\mathbf{S} = \mathbf{S}_1 * \mathbf{n} + \mathbf{S}_2 * \mathbf{n} + \mathbf{S}_3 * \mathbf{n}. \quad (4)$$

Converting S back to polar co-ordinates (S, t', p') gives its orientation point **n**′. Let reciprocal principal stretch vectors S_1', S_2', S_3', be parallel to the principal stretches S_1, S_2, S_3, but have inverse magnitudes $1/S_1, 1/S_2, 1/S_3$. Then the reciprocal stretch vector **S**′ is given by

$$\mathbf{S}' = \mathbf{S}_1' * \mathbf{n} + \mathbf{S}_2' * \mathbf{n} + \mathbf{S}_3' * \mathbf{n}. \quad (5)$$

Its trend and plunge (t'', p''), is the same as that of pole **n**″ in Fig. 1(b). The angle between vectors **n**′ and **n**″ is given by the standard dot product method, that is, $\cos \psi = \mathbf{S} \cdot \mathbf{S}' / |\mathbf{S}||\mathbf{S}'|$. ψ is the required angular shear.

As stated in the previous section, the deformed direction of a line is more commonly known in structural geology. Since the graphical construction involved a double application of the reverse deformation, the equivalent algebraic solution is to substitute vectors $\lambda_1', \lambda_2', \lambda_3'$ into equation (1), where each $\lambda' = 1/S^2$,

$$\lambda' = \lambda_1' * \mathbf{n}' + \lambda_2' * \mathbf{n}' + \lambda_3' * \mathbf{n}'. \quad (6)$$

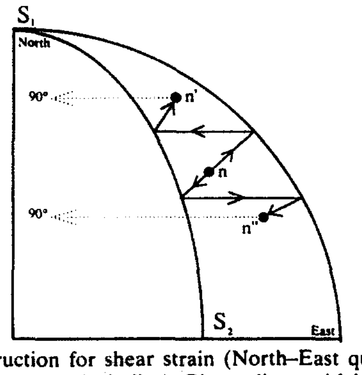

Fig. 4. Construction for shear strain (North–East quadrant shown; North–Down quadrant is similar). Given a line **n** with initial trend t and plunge p, its deformed direction $\mathbf{n}' = (t', p')$ is located as in Fig. 2(b). The point $\mathbf{n}'' = (t'', p'')$ is located by reversing the procedure—it represents the pole to a plane that was perpendicular to **n** before deformation. The angle between **n**″ and **n**′ represents the angular shear ψ of the initial line **n** or final line **n**′.

Again, the trend and plunge of λ' give the direction of pole $\mathbf{n}''$ and the angular shear, $\psi = \mathbf{n}' \wedge \mathbf{n}''$, may be obtained using the dot product.

CONCLUSIONS

The graphical constructions presented here may appear difficult due to lack of familiarity, but they are as easy to master as the three-dimensional Mohr circle constructions which give the same results. The algebraic derivation presented here averts the need to introduce strain tensors by employing a simple vector product, $\mathbf{a}*\mathbf{b}$. Of course, the simplicity of equations (1)–(6) is of benefit only if they can be solved in practice. To that end, a computer code is included for the solution of equations (1)–(6).

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APPENDIX

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BASIC Computer Algorithms.
Pi=ATN(1)<<2
Radians=Pi/180
Degrees=180/Pi

FOR j=1 TO 3
  INPUT "Principal vector ";R,Theta,Phi
  D(1,j),D(2,j),D(3,j)=FN CARTESIAN(1,Theta,Phi)
  P(1,j),P(2,j),P(3,j)=FN CARTESIAN (R,Theta,Phi)
  R(j)=R
NEXT j

INPUT "Trend,plunge of pole"; Theta, Phi
n(1),n(2),n(3)=FN CARTESIAN(1,Theta,Phi)

FOR i=1 TO 3, j=1 TO 3, k=1 TO 3
  St(i)=St(i)+D(i,j)*n(k)*P(k,j)          'eqn (1)
NEXT k, j, i

Dot=n(1)*St(1)+n(2)*St(2)+n(3)*St(3)

FOR i=1 TO 3
  Sn(i)=n(i)*Dot                          'eqn (2)
  Ss(i)=St(i)-Sn(i)                       'eqn (3)
  S(i)=St(i)                              'eqn (4)
NEXT i

FOR j=1 TO 3,i=1 TO 3
  IF R(j)>0
    s(i,j)=D(i,j)/R(j)                    'reciprocals
    l(i,j)=D(i,j)/R(j)^2       'reciprocal quadratics
  END IF
NEXT i,j

FOR i=1 TO 3, j=1 TO 3, k=1 TO 3
  s(i)=s(i)+D(i,j)*n(k)*s(k,j)            'eqn (5)
  l(i)=l(i)+D(i,j)*n(k)*l(k,j)            'eqn(6)
NEXT k, j, i

INPUT "Enter option";Datatype$

SELECT Datatype$
  CASE "stress on plane"
    FN POLAR ("Total stress",St(1),St(2),St(3))
    FN POLAR ("Normal stress",Sn(1),Sn(2),Sn(3))
    FN POLAR ("Shear stress",Ss(1),Ss(2),Ss(3))
  CASE "shear of initial line"
    PRINT "Angular shear of initial line n=";
    FN ANGLE(S(1),S(2),S(3),s(1),s(2),s(3))
  CASE "shear of final line"
    PRINT "Angular shear of final line n'=";
    FN ANGLE(l(1),l(2),l(3),n(1),n(2),n(3))
END SELECT

END

Theta=FN ARCTAN(y,x)
  IF x=0
    Theta=90
  XELSE
    Theta=ATN(ABS(y/x))*Degrees
    IF x<0 THEN Theta=180-Theta
  END IF
  IF y<0 THEN Theta=360-Theta
END FN                    'range (0°,360°) for trends

Phi=FN ARCSIN(x,y,z)
  IF z<0
    x=-x
    y=-y
    z=-z
  END IF
  Phi=FN ARCTAN(z,SQR(1-z^2))
END FN                    'range (0°,90°) for plunges

Psi=FN ARCCOS(z)
  Psi=FN ARCTAN(SQR(1-z^2),z)
END FN                    'range (0°,180°) for pitches

x,y,z=FN CARTESIAN(R,Theta,Phi)
  Theta=Theta*Radians
  Phi=Phi*Radians
  x=R*COS(Theta)*COS(Phi)
  y=R*SIN(Theta)*COS(Phi)
  z=R*SIN(Phi)
END FN

FN POLAR(Title$,x,y,z)
  R=SQR(x^2+y^2+z^2)
  IF R >0
    x=x/R
    y=y/R
    z=z/R
    Phi=FN ARCSIN(x,y,z)
    Theta=FN ARCTAN(y,x)
  XELSE
    Phi=0
    Theta=0
  END IF
  PRINT Title$,R,Theta,Phi
END FN

FN ANGLE(A(1),A(2),A(3),B(1),B(2),B(3))
MagA=SQR(A(1)^2+A(2)^2+A(3)^2)
MagB=SQR(B(1)^2+B(2)^2+B(3)^2)
Denominator=MagA*MagB
Dot=A(1)*B(1)+A(2)*B(2)+A(3)*B(3)
Argument=Dot/Denominator
Psi=FN ARCCOS(Argument)
PRINT Psi
END FN
```